USING UNCERTAINTY TO ESTABLISH CERTAINTY

Po-Shen Loh, Princeton University

IPAM Fall 2009

Combinatorics:

Methods and Applications in

Mathematics and Computer Science

Sociology

Observation (S. Szalai, sociologist)

Every group of about 20 children contains a set of 4 children, any two of which are friends, or a set of 4 children, no two of which are friends.

Sociology . . . or Ramsey Theory?

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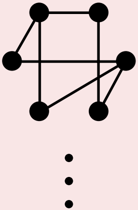
. . . but after discussion with Hungarian mathematicians Erdős, Turán, and Sós:

Ramsey number $R(4, 4)$



Draw 18 points, and connect some pairs of them by lines. No matter how this is done, there will always exist either:

- a set of 4 points, with all pairs connected, or
- a set of 4 points, with no pairs connected.

Definition

Let $R(r, s)$ be the smallest integer such that every graph with $R(r, s)$ vertices contains either a clique of size r or an independent set of size s.

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Proof. Induction on $r+s$. Let $u_{r,s} = \binom{r+s-2}{r-1}$.
Since $u_{r,s} = u_{r-1,s} + u_{r,s-1}$, any vertex $v \in G$ has either:

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- at least $u_{r,s-1}$ non-nbrs. $\Rightarrow$ independent set of size $s-1$. $\square$

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- The diagonal bound is $R(r,r) \le \binom{2r-2}{r-1} \approx 2^{2r}$.
- To lower-bound $R(r,r)$, one must construct a large graph with all cliques and independent sets smaller than r.
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Proof.

- Let $n = 2^{r/2}$, and let $V = \{v_1, \ldots, v_n\}$ be vertices.
- For each pair of vertices, place an edge with probability $\frac{1}{2}$.
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- For each of the $\binom{n}{r}$ sets S, $\mathbb{P}[B_S] = 2 \cdot 2^{-\binom{r}{2}}$. So $\mathbb{P}[\text{some } B_S \text{ occurs}]$ is at most

$$\begin{aligned} \binom{n}{r} \cdot 2 \cdot 2^{-\binom{r}{2}} &\leq \frac{n^r}{r!} \cdot 2 \cdot 2^{-\frac{r^2-r}{2}} \\ &= (2^{r/2})^r / r! \cdot 2^{1-\frac{r^2-r}{2}} \\ &= 2^{1+r/2}/r! < 1. \quad \square \end{aligned}$$

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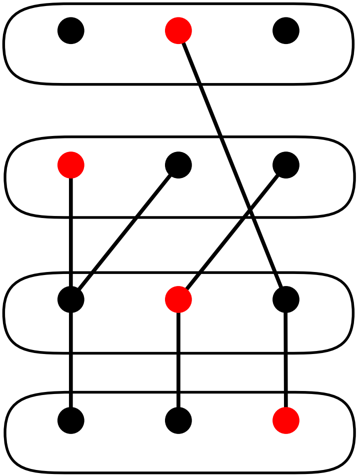
Independent transversals

Definition

An *independent transversal* has one vertex per group, with no edges between the vertices.



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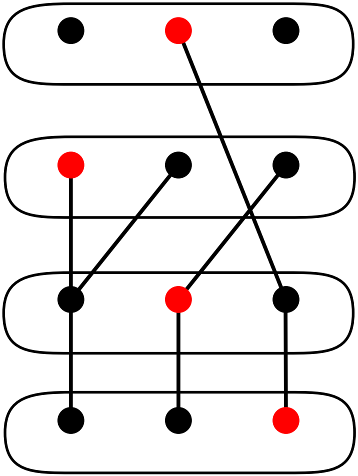

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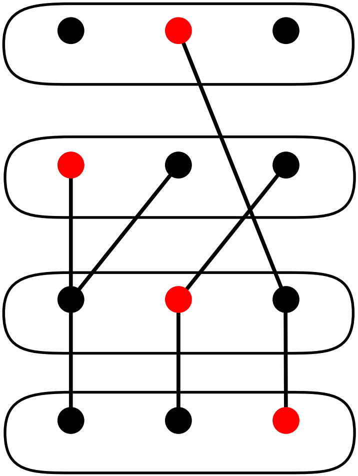
If every group has size $\geq 2e\Delta$, then indep. trans. always exists, *no matter how many groups there are.*



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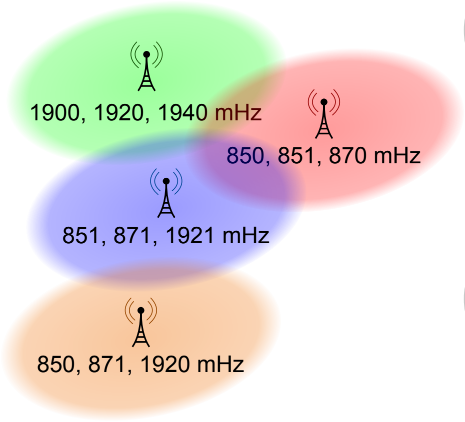

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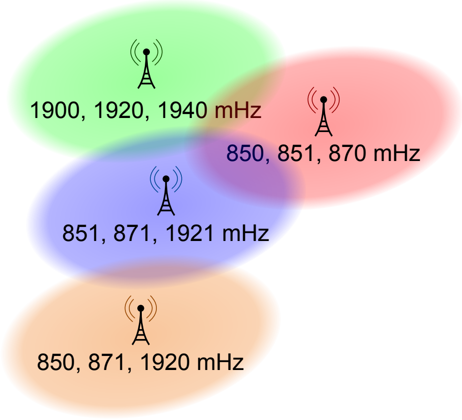

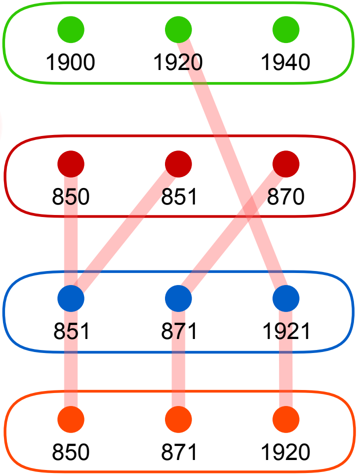

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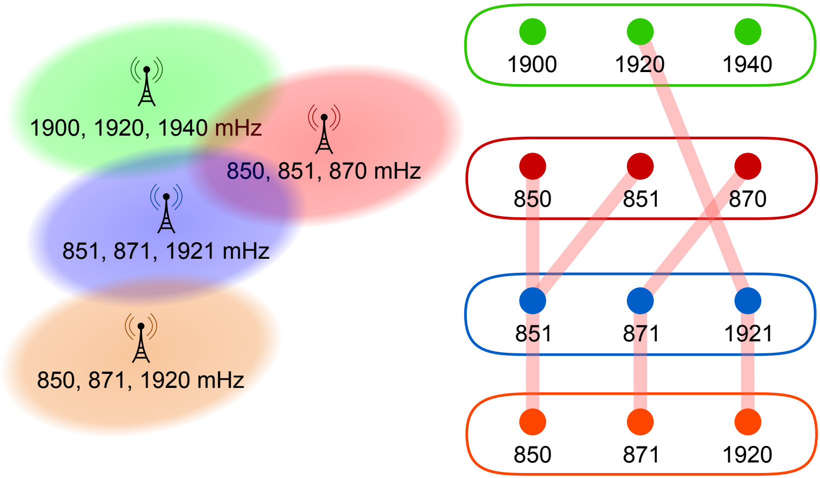

Remark. In this example, Δ is bounded by local geometry, but the number of towers (vertex groups) can be arbitrarily large.

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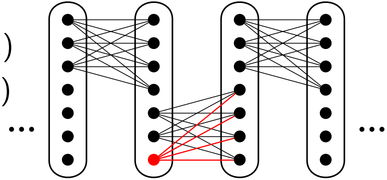
  Construction with sizes exactly $2\Delta - 1$, *but no indep. trans.*



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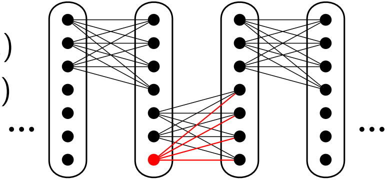

- But if degrees are not concentrated,* then sizes $\geq (1 + o(1))\Delta$ suffice. (L.-Sudakov, 2007)

  * *i.e., if each vertex sends only $o(\Delta)$ edges into each other part*

Question

Let $B_1, \dots, B_n$ be "bad" events in a probability space. How can one show that with positive probability, none of the B_i occur?

Observations:

- For the Ramsey lower bound, the union bound $\mathbb{P}[\text{some } B_i] \leq \sum \mathbb{P}[B_i]$ was already below 1.

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- Consider flipping 2000 fair coins, and let B_i be the event that the i-th coin is heads.
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- The union bound only gives $\mathbb{P}[\text{some } B_i] \leq \sum \mathbb{P}[B_i] = 1000$.
- Yet no matter how many *independent* coins we flip, it is possible (although unlikely) that all are tails.

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Let $B_1, \ldots, B_n$ be "bad" events, such that for some p, d:

- Every $\mathbb{P}[B_i] \leq p$.
- Each B_i is independent of all but $\leq d$ of the other B_j.
- $ep(d+1) \leq 1$, where $e \approx 2.718$.

Then with positive probability, none of the B_i occur.

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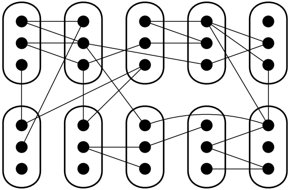

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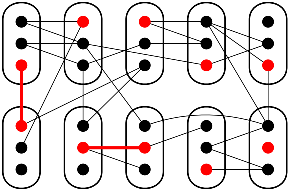

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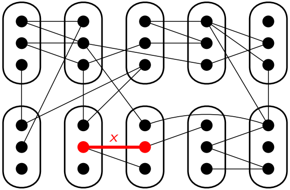

The Local Lemma

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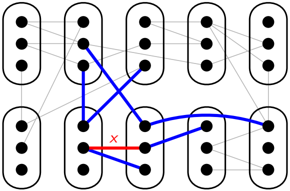

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- For each edge x, let B_x be the event that both endpoints of x were picked.
- Let $p = \mathbb{P}[B_x] = \frac{1}{(2e\Delta)^2}$.
- Let $d = 2 \cdot (2e\Delta) \cdot \Delta - 2$.
- Then $ep(d+1) < 1$, so there is an outcome when none of the B_x occur, i.e., an independent transversal exists. $\square$

Sperner's Theorem

Sperner (1928)

Let $\mathcal{F}$ be a family of subsets of $\{1, \ldots, n\}$ that is an *antichain*, i.e., no $A, B \in F$ satisfy $A \subset B$. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

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Proof that $\displaystyle \sum_{\mathbf{A} \in \mathcal{F}} \frac{\mathbf{1}}{\binom{\mathbf{n}}{|\mathbf{A}|}} \leq \mathbf{1}.$

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Proof that $\displaystyle \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1.$

- Let $(x_1, x_2, \ldots, x_n)$ be a random permutation of $\{1, \ldots n\}$.
- For each $A \in \mathcal{F}$, let E_A be the event that $\{x_1, \ldots, x_{|A|}\} = A$.

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- For each $A \in \mathcal{F}$, let E_A be the event that $\{x_1, \dots, x_{|A|}\} = A$.
- The E_A are mutually exclusive since $\mathcal{F}$ is an antichain, so:

$$\sum_{A\in\mathcal{F}} \mathbb{P}[E_A] \;=\; \mathbb{P}[\text{some } E_A \text{ occurs}] \;\le\; 1.$$

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- The E_A are mutually exclusive since $\mathcal{F}$ is an antichain, so:

$$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} = \sum_{A \in \mathcal{F}} \mathbb{P}[E_A] = \mathbb{P}[\text{some } E_A \text{ occurs}] \leq 1.$$

$\square$

Erdős (1945)

Let $x_1, \ldots, x_n$ be real numbers greater than 1. Let S be a collection of sums of distinct x_i, such that any $s, s' \in S$ satisfy $|s - s'| \leq 1$. Then $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.

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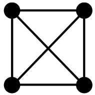
Proof.

- For each element $s \in S$, we may define a set $A_s \subset \{1, \ldots, n\}$ such that $s = \sum_{i \in A_s} x_i$.
- Let $\mathcal{F}$ be the collection of all such A_s.
- Every $A_s \not\subset A_{s'}$ because all $x_i > 1$.
- Sperner's Theorem implies that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. □

Definition

A graph is *planar* if it can be drawn with no crossing edges.



K_4

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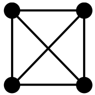

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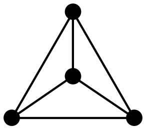

K_4, planar drawing

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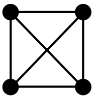

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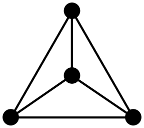

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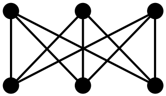

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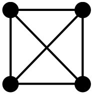

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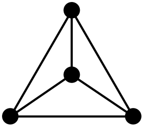

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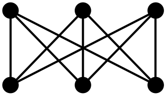

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Famous theorems:

- The vertices of any planar graph can be colored with only 4 colors, s.t. no pair of adjacent vertices gets the same color.
- Kuratowski: A graph is planar iff it does not contain a topological copy of $K_{3,3}$ or K_5.
- Euler formula: $Vertices - Edges + Faces = 2$.

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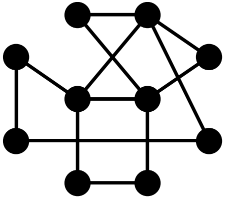

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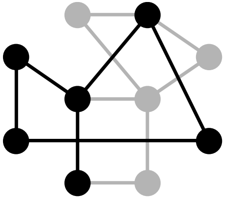

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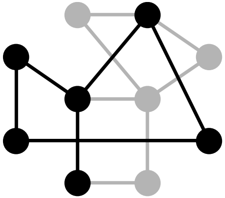

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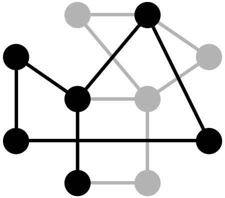

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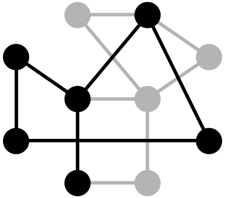

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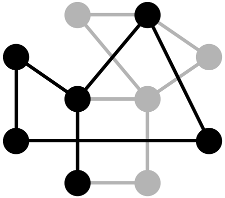

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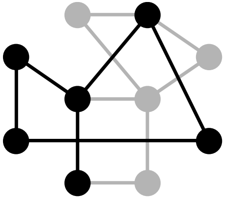

$$\begin{aligned} Xp^4 &\geq Ep^2 - 3Vp \\ X &\geq p^{-2} \cdot (E - 3Vp^{-1}) \\ &= \left(\frac{E}{4V}\right)^2 \cdot \frac{E}{4}. \end{aligned}$$

□

Point-line incidences

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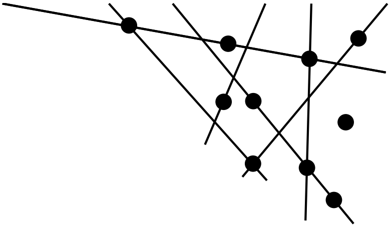
Let P be a set of n points, and L be a set of m lines. Then only $I \leq 4(m^{2/3}n^{2/3} + m + n)$ pairs $(p, \ell) \in P \times L$ have p lying on ℓ.



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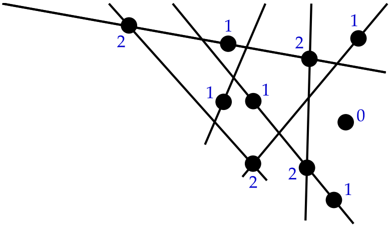

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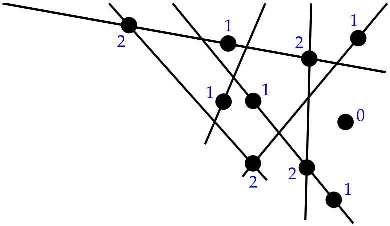

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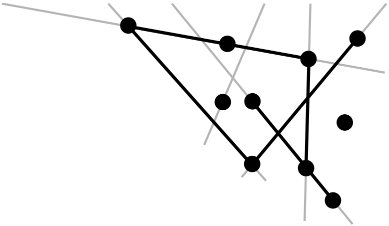

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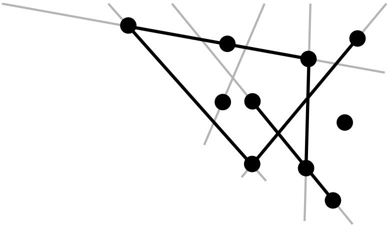

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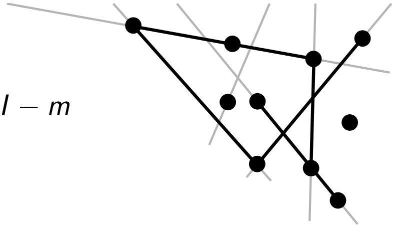

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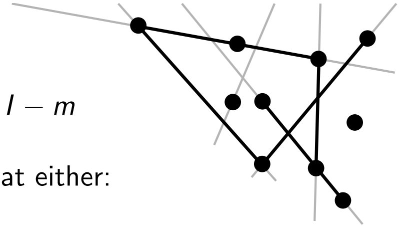

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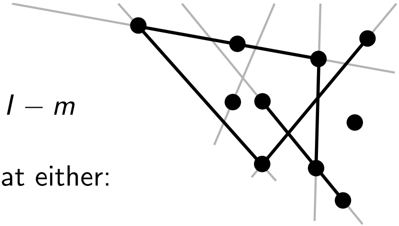

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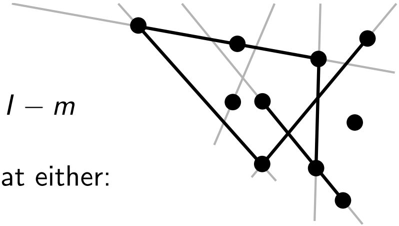

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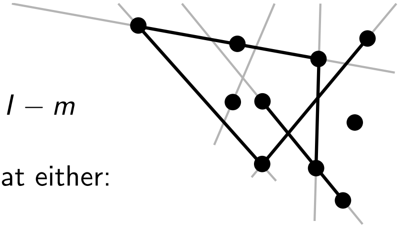

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Both cases give $I - m \leq 4(m^{2/3}n^{2/3} + n)$. □

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For $A \subset \mathbb{R}$, let $A + A = \{a + b : a, b \in A\}$, $A \cdot A = \{ab : a, b \in A\}$.

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Combinatorial number theory

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- $c = \frac{3}{11}$, Solymosi (2005)
- $c = \frac{1}{3} - \epsilon$, Solymosi (2008)

Proof that $|A + A|$ or $|A \cdot A|$ is always $\gtrsim n^{5/4}$, when $|A| = n$.

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$$n^{9/3} \quad = \quad n^3 \quad \leq \quad I \quad \leq \quad 4 \cdot 3 \cdot |L|^{2/3}|P|^{2/3} \quad = \quad 12 \cdot n^{4/3} \cdot |P|^{2/3}$$

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$$
\begin{aligned}
n^{9/3} = n^3 \leq I &\leq 4 \cdot 3 \cdot |L|^{2/3}|P|^{2/3} = 12 \cdot n^{4/3} \cdot |P|^{2/3} \\
\frac{1}{12} n^{5/3} &\leq |P|^{2/3} \\
0.024 n^{5/2} &\leq |P|
\end{aligned}
$$

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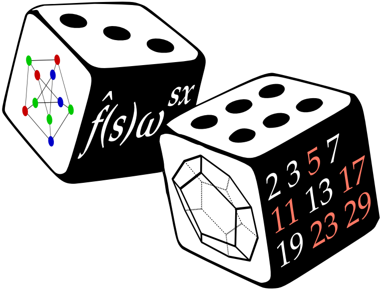
$$\begin{aligned} n^{9/3} = n^3 \leq I &\leq 4 \cdot 3 \cdot |L|^{2/3}|P|^{2/3} = 12 \cdot n^{4/3} \cdot |P|^{2/3} \\ \frac{1}{12}n^{5/3} &\leq |P|^{2/3} \\ 0.024n^{5/2} &\leq |P| = |A + A| \cdot |A \cdot A|. \end{aligned}$$

Therefore, $|A + A|$ or $|A \cdot A|$ must be $\gtrsim n^{5/4}$. $\square$



IPAM Fall 2009

Los Angeles, California

Combinatorics:

Methods and Applications in

Mathematics and Computer Science

Workshop 1. Probabilistic techniques and applications

Workshop 2. Combinatorial geometry

Workshop 3. Topics in graphs and hypergraphs

Workshop 4. Analytical methods in combinatorics, additive number theory and computer science

Organizers: *N. Alon, G. Kalai, J. Pach, V. Sós, A. Steger, B. Sudakov, T. Tao.*